TAME HOMOMORPHISMS OF POLYTOPAL RINGS

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Abstract

The object of this paper is the tameness conjecture introduced in [3] which describes an arbitrary graded k-algebra homomorphism of polytopal rings. We will give further evidence of this conjecture by showing supporting results concerning joins, multiples and products of polytopes.

1 Introduction

This paper studies the category of polytopal algebras over a field k, denoted Pol(k). In particular we investigate the concept of *tameness*, as introduced in [3] where it is conjectured that any graded k-algebra homomorphism is tame. In short, such a homomorphism is called tame if it can be obtained by a composition of some standard ones, defined in [3] and explained below.

In this work we will show the following. Given that any graded homomorphism between two polytopal rings is tame, then any graded homomorphisms from the rings obtained by taking multiples, joins, and products of the underlying polytopes are also tame. Thus we extend the class of tame homomorphisms, giving further evidence in support of the mentioned conjecture. The main result of this work is the result concerning products of polytopes. The result regarding joins mainly follows from well-known properties of this operation, while to prove the results concerning multiples and, especially, products require more involved arguments.

The objects of Pol(k), the *polytopal (monoid) rings*, are defined as follows. Let P be a convex lattice polytope in $\mathbb{R}^n$. Let $\mathrm{L}(P)$ denote the lattice points in P, i.e. $\mathrm{L}(P) = P \cap \mathbb{Z}^n$, and let $\mathrm{S}(P)$ be the additive monoid of $\mathbb{Z}^{n+1}$ generated by $\{(x,1) \,|\, x \in \mathrm{L}(P)\}$. A lattice point in $\mathrm{S}(P)$ can in a natural way be represented as a monomial in $n+1$ variables by identifying its coordinate vector with the monomial's exponent vector. By the degree of a monomial we will mean the last component of its exponent vector. The polytopal ring $k[P]$ is the monoid ring of $\mathrm{S}(P)$ with coefficients in k. $k[P]$ is a graded ring and generated by its degree 1 monomials, which correspond bijectively to the lattice points in P. The generators' relations are the binomial relations representing the affine dependencies in $\mathrm{L}(P)$.

A homomorphism in Pol(k) is a homomorphism of two polytopal rings as k-algebras that preserves the grading. Such a homomorphism has to also preserve the binomial relationships among the generators. It follows that $\mathrm{Hom}(k[P], k[Q])$, the set of all graded homomorphism between the two polytopal rings, is the zero set of a system of polynomials, and hence gives rise to a Zariski closed set in the space of matrices $M_{mn}(k)$, where $m = \#L(Q)$ and $n = \#L(P)$, which we call the *Hom-variety*. We wish to be able to describe the structure of this set and the tameness conjecture, explained next, is a geometric description of the Hom-variety.

A tame homomorphism is a graded k-algebra homomorphism that can be obtained by a composition of four standard homomorphisms, called polytope changes, homothetic blow-ups, Minkowski sums, and free extensions. Assume $f : k[P] \to k[Q]$ is a graded homomorphism and $P, Q \subset \mathbb{R}^n$ are lattice polytopes.

Suppose we are given a subpolytope $P' \subset P$ and a polytope $Q' \supset Q$ such that $f(k[P']) \subset k[Q']$. Then f gives rise to a new homomorphism $f' : k[P'] \to k[Q']$ in a natural way. Also, if we have two polytopes

$\tilde{P}$ and $\tilde{Q}$ that are isomorphic to P respectively Q as lattice polytopes, then f induces a homomorphism $\tilde{f} : k[\tilde{P}] \to k[\tilde{Q}]$. The homomorphisms obtained this way are called *polytope changes*.

Consider the normalization of $S(P)$, defined by

$$\overline{S(P)} = \{x \in \mathrm{gp}(S(P)) \,|\, x^m \in S(P) \text{ for some } m \in \mathbb{N}\}$$

and the normalization of $S(Q)$, denoted $\overline{S(Q)}$ and defined similarly. Here $\mathrm{gp}(S(P))$ denotes the group of differences of $S(P)$. $S(P)$ is normal if and only if $S(P) = \overline{S(P)}$. It is well known that $k[\overline{P}] := k[\overline{S(P)}] = \overline{k[P]}$. Suppose there are no monomials in the kernel of f, that is $Ker(f) \cap S(P) = \emptyset$. Then f extends uniquely to the homomorphism $\overline{f} : k[\overline{P}] \to k[\overline{Q}]$ defined by

$$\overline{f}(x) = \frac{f(y)}{f(z)} \quad \text{where } \; x = \frac{y}{z},\ x \in \overline{S(P)}, \text{ and } y, z \in S(P).$$

(That $f(y)/f(z)$ in fact belongs to $k[\overline{Q}]$ follows from the fact that $x \in \overline{S(P)}$ implies $x^c \in S(P)$, for some natural number c, which in turn implies $f(x^c) = (f(y)/f(z))^c \in k[Q]$, i.e. $f(y)/f(z) \in k[\overline{Q}]$) . Let $k[\overline{P}]_c$, $c \in \mathbb{N}$, denote the subring of $k[\overline{P}]$ generated by the homogeneous components of degree c in $\overline{S(P)}$, and similarly for $k[\overline{Q}]_c$. Then $k[\overline{P}]_c \simeq k[cP]$ and $k[\overline{Q}]_c \simeq k[cQ]$ in a natural way. Since f is graded, $\overline{f}$ restricts to the graded homomorphism $f^{(c)} : k[cP] \to k[cQ]$, which we call the *homothetic blow-up* of f.

Now consider the case when we have two graded homomorphisms $f, g : k[P] \to k[Q]$. Let $\mathrm{N}(f(x))$ denote the Newton polytope of $f(x)$, i.e. the convex hull of the support monomials in $f(x)$, and similarly for $\mathrm{N}(g(x))$. Assume f and g satisfy

$$\mathrm{N}(f(x)) + \mathrm{N}(g(x)) \subset Q \quad \forall x \in L(P)$$

where $+$ denotes the Minkowski sum in $\mathbb{R}^n$. Then $z^{-1}f(x)g(x) \in k[Q]$ where $z = (0, 0, ..., 1)$, and thus

$$f \star g : k[P] \to k[Q] \;\text{ such that }\; f \star g(x) = z^{-1}f(x)g(x)$$

for all $x \in L(P)$ defines a new graded homomorphism. This process of obtaining new homomorphisms is called *Minkowski sums*.

The last standard homomorphism is obtained as follows. Assume P is a pyramid with base P_0 and vertex v such that $L(P) = \{v\} \cup L(P_0)$ and that we are given a homomorphism $f_0 : k[P_0] \to k[Q]$. Then $k[P]$ is a polynomial extension of $k[P_0]$ and f_0 extends to a homomorphism $f : k[P] \to k[Q]$ by letting $f(v) = q$ for any $q \in k[Q]$ and $f(x) = f_0(x)$ for all $x \in \mathrm{L}(P_0)$. This process is called *free extensions*.

The tameness conjecture, as found in [3], states the following.

Conjecture 1. (W. Bruns, J. Gubeladze) *Any homomorphism in* $\mathrm{Pol}(k)$ *is obtained by a sequence of taking free extensions, Minkowski sums, homothetic blow-ups, polytope changes and compositions, starting from the identity mapping $k \to k$. Moreover, there are normal forms of such sequences.*

Certain subvarieties of the Hom-variety have been described. In [1] the subvariety corresponding to automorphisms has been described completely. This result shows that any automorphism is a composition of some basic ones and can be viewed as a polytopal generalization of the linear algebra fact that any invertible matrix can be written as the product of elementary, permutation, and diagonal matrices. The result is stronger than the notion of tameness, thus implying that any automorphism is tame.

In [2] the variety corresponding to codimension-1 retractions of polygons has been described, and it is conjectured that the result generalizes to arbitrary dimensions of the polytope. This result is also stronger than tameness and hence any such retraction is tame.

In showing that the above results imply tameness of the corresponding morphisms some other classes of homomorphisms were shown to be tame in [3]. Among those we will use that homomorphisms respecting monomial structures and any homomorphism from $k[c\Delta_n]$ (where Δ_n is the n-simplex, c, $n \in \mathbb{N}$) are tame. We will also use the notion of face retractions, which are idempotent endomorphisms of $k[P]$ defined by

$\pi_F(x) = \begin{cases} x, & \text{if } x \in F \\ 0, & \text{if } x \notin F \end{cases}$ for all $x \in L(P)$, where F is a face of P (note that face retractions are tame since they respect the monomial structures).

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2 Main Results

The results of this work concern joins, multiples, and Segre products.

The join of two lattice polytopes $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ of dimension n respectively m is a subset of $\mathbb{R}^{n+m+1}$ obtained as follows.
First consider the embeddings ι_1 and ι_2 of $\mathbb{R}^n$ respectively $\mathbb{R}^m$ into $\mathbb{R}^{n+m+1}$ defined as

$$\iota_1 : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0) \qquad \forall\, (x_1, \dots, x_n) \in \mathbb{R}^n$$

$$\iota_2 : (y_1, \dots, y_m) \mapsto (0, \dots, 0, y_1, \dots, y_m, 1) \quad \forall\, (y_1, \dots, y_m) \in \mathbb{R}^m$$

We define the join of P and Q as the convex hull of the image of P under ι_1 and the image of Q under ι_2, i.e. $\mathrm{join}(P, Q) = \mathrm{conv}(\iota_1(P), \iota_2(Q))$.

We denote the c^{th} multiple of P by cP, i.e. $cP = \{cx \mid x \in P\}$. Note that in general $k[cP]$ is an overring of the c^{th} Veronese subring of $k[P]$ (the ring generated by the homogeneous components of degree c in $S(P)$), since the lattice points in $S(P)$ sitting on height c in general corresponds to a subset of the lattice points in cP. The two rings coincide when P is normal.

The Segre product of $k[P]$ and $k[Q]$ is $k[P \times Q]$ where $P \times Q = \{(x, y) \mid x \in P, y \in Q\}$.

Theorem 1. *Suppose any graded homomorphism from $k[P]$ respectively $k[Q]$ is tame. Then*

(a) any graded homomorphism from $k[\mathrm{join}(P, Q)]$ is tame.

(b) any graded homomorphism from $k[cP]$, where $c \in \mathbb{N}$, is tame.

(c) any graded homomorphism from $k[P \times Q]$ is tame.

The proof of this theorem is found in sections 4, 5, and 6. For all three parts we will use the following lemma, which we here state and prove in the case of Segre products, but similar arguments hold for joins and multiples.

Lemma 1. *Let f be a graded homomorphism from $k[P \times Q]$ and assume the hypothesis of Theorem 1. To show that f is tame we can without loss of generality assume that $\ker(f) \cap S(P \times Q) = \emptyset$.*

Proof. Let $f : k[P \times Q] \to k[R]$ be a graded homomorphism (where R is some lattice polytope). Assume there exist a monomial $m \in \ker(f)$. Since $\ker(f)$ is a prime ideal, the ideal $I = (\ker(f) \cap S(P \times Q) \subset k[P \times Q]$ is a monomial prime ideal containing m. However, monomial prime ideals are exactly the kernels of face retractions [4].

Thus f factors through a face retraction (known to be tame) and a map $g : k[F] \to k[R]$ where F is a face of $P \times Q$. A face of $P \times Q$ is of the form $P' \times Q'$ where P' is a face of P and Q' a face of Q [6].

Note that any homomorphism $f' : k[P'] \to k[R]$ is tame since it factors through

$$k[P'] \xrightarrow{x \mapsto x} k[P] \xrightarrow{x \to \pi_{P'}(x)} k[P'] \xrightarrow{x \mapsto f'(x)} k[R]$$

where the first map is tame since it maps monomials to monomials and the composition of the last two maps is tame since it is a homomorphism from $k[P]$ (tame by assumption). Similarly any homomorphisms from $k[Q']$ is tame. Thus Theorem 1(c) is proved once we have shown tameness of any homomorphism f' from $k[P' \times Q']$ such that $\ker(f') \cap S(P' \times Q') = \emptyset$, given that any homomorphisms from $k[P']$ and $k[Q']$ is tame. $\square$

3 Translation into Discrete Objects

The following is a description of how a graded homomorphism in $\mathrm{Hom}(k[P], k[Q])$ can be viewed as an affine integral map, and vice versa, and in this way we can translate polynomials in a polytopal ring into discrete objects. Here we will assume that there are no monomials in the kernel. This translation will aid in proving that certain homomorphisms are tame.

Suppose P and Q are lattice polytopes such that $\mathrm{L}(P) \subset \mathbb{Z}_+^d$ and $\mathrm{L}(Q) \subset \mathbb{Z}_+^e$. This assumption will not affect the tameness of a homomorphism between $k[P]$ and $k[Q]$ since for any $P \subset \mathbb{R}^d$ we can assume $\mathrm{L}(P) \subset \mathbb{Z}_+^d$ by a polytope change.
Let $f : k[P] \to k[Q]$ be a graded homomorphism. Since $\mathrm{L}(Q) \subset \mathbb{Z}_+^e$, $\mathrm{L}(Q) \subset \{X_1^{a_1} \cdots X_e^{a_e} Y | a_i \in \mathbb{Z}_+\}$ and thus the polynomials $\varphi_x = f(x)Y^{-1}$ belong to $k[X_1, ..., X_e]$ for all $x \in L(P)$. Clearly, since f respects the binomial relations in $L(P)$, so do the polynomials φ_x.

Now $k[X_1, ..., X_e]$ is a unique factorization domain and therefore $k[X_1, ..., X_e] \setminus \{0\}/k^*$ is a free commutative monoid. Thus there is a subset $\mathcal{P} \subset k[X_1, ..., X_e] \setminus \{0\}/k^*$ of irreducibles such that for each $[\varphi_x] \in k[X_1, ..., X_e] \setminus \{0\}/k^*$ we have $[\varphi_x] = P_1^{a_1} \cdots P_l^{a_l}$ for some $P_i \in \mathcal{P}$, $a_i \geq 0$, $l \in \mathbb{N}$. But since only finitely many irreducibles are needed to represent any $[\varphi_x]$ there exists an $l \in \mathbb{N}$ such that $[\varphi_x] \in \{P_1^{a_1} \cdots P_l^{a_l} | P_i \in \mathcal{P}, a_i \in \mathbb{Z}_+\}$ for all $x \in \mathrm{L}(P)$. Note that $\{P_1^{a_1} \cdots P_l^{a_l} | P_i \in \mathcal{P}, a_i \in \mathbb{Z}_+\}$ is isomorphic to $\mathbb{Z}_+^l$ and thus each polynomial $f(x)$, with $x \in L(P)$, gives rise to an integral vector by the correspondence

$$f(x) \mapsto \ [\varphi_x] = P_1^{a_1} \cdots P_l^{a_l} \mapsto (a_1, \ldots, a_l)$$

Thus f gives rise to an affine map from $\mathrm{L}(P)$ to $\mathbb{Z}_+^l$ which respects the binomial relations in $\mathrm{L}(P)$.

Conversely suppose we are given the affine map $\alpha : \mathrm{L}(P) \to \mathbb{Z}_+^l$. Then for each $x \in \mathrm{L}(P)$, $\alpha(x) \in \mathbb{Z}_+^l$ gives rise to a polynomial p_x in $k[X_1, \ldots, X_e]$ under the correspondence

$$\alpha(x) = (a_1, \ldots, a_l) \mapsto P_1^{a_1} \cdots P_l^{a_l} = p_x.$$

Then $\varphi_x = t_x p_x$ for some nonzero scalar t_x for all $x \in L(P)$. Normalizing by letting $t_x p_x \mapsto p_x$ for all $x \in L(P)$, we have $f(x) = p_x Y$. Thus we recover the homomorphism f from α.

Also note that we can extend α to an affine integral map $P \to \mathbb{R}_+^l$ in the following way. If $\mathrm{L}(P) = \{x_0, \ldots, x_n\}$ then $P = \mathrm{conv}(x_0, \ldots, x_n)$ and thus any $x \in P$ can be represented as $x = c_0 x_0 + \cdots + c_n x_n$ for some nonnegative scalars c_i such that $\sum_{i=0}^n c_i = 1$. Thus if $f(x_i)$ corresponds to $a_i \in \mathbb{Z}_+^l$ for each $i \in [0, n]$, $f(x)$ corresponds to $c_0 a_0 + \cdots + c_n a_n \in \mathbb{R}_+^l$. This correspondence is well-defined since f respects the binomial relations in $k[P]$. Conversely, if $\alpha(x_i)$ corresponds to $p_{x_i} \in k[X_1, \ldots, X_e]$, then $f(x) = p_{x_0}^{c_0} \cdots p_{x_n}^{c_n} Y$.

4 Joins

The proof of Theorem $1(a)$ mainly follows from the following well-known properties of joins.

It is known that the relations among the lattice points in $\mathrm{join}(P, Q)$ comes only from the relations in $\mathrm{L}(P)$ and $\mathrm{L}(Q)$, i.e. there are no relations between the lattice points coming from P and the lattice points coming from Q, and thus any two polytopal rings $k[P]$ and $k[Q]$ satisfy $k[\mathrm{join}(P, Q)] \simeq k[P] \otimes k[Q]$. Also note that, by definition, there are no new lattice points in $\mathrm{join}(P, Q)$ since if $x \in \mathrm{L}(\mathrm{join}(P, Q))$ then $x \in \mathrm{L}(\iota_1(P)) \simeq \mathrm{L}(P)$ or $x \in \iota_2(\mathrm{L}(Q)) \simeq \mathrm{L}(Q)$.

Thus any graded homomorphisms $f : k[P] \to k[L]$ and $g : k[Q] \to k[L]$ defines a new graded homomorphism $F : k[\mathrm{join}(P, Q)] \to k[L]$ by letting

$$F(x) = \begin{cases} f(x), & \text{if } x \in \mathrm{L}(P) \\ g(x) & \text{if } x \in \mathrm{L}(Q) \end{cases} \qquad \forall x \in \mathrm{L}(\mathrm{join}(P, Q))$$

and conversely any homomorphism $F : k[\mathrm{join}(P, Q)] \to k[L]$ is necessarily of this form.

Thus Theorem $1(a)$ is proved once we have shown that this "pasting" of the two tame homomorphisms is also tame.

Proof of Theorem 1(a). Suppose P and Q are lattice polytopes. Let $F : k[\mathrm{join}(P,Q)] \to k[R]$ be any graded homomorphism where R is a lattice polytope in $\mathbb{R}^d$. By a polytope change we may assume $Z = (0,\dots,0,1) \in \mathrm{L}(R)$. Assume $\mathrm{L}(P) = \{x_0,\dots,x_n\}$ and $\mathrm{L}(Q) = \{y_0,\dots,y_m\}$. By the above argument F is defined by
$F(x_i) = f(x_i)$ for $i \in [0,n]$ for some homomorphism $f : k[P] \to k[R]$
$F(y_j) = g(y_j)$ for $j \in [0,m]$ for some homomorphism $g : k[Q] \to k[R]$.
Now consider the homomorphisms

$$a_P : k[\mathrm{join}(P,Q)] \to k[R] \text{ such that } a_P(x_i) = f(x_i),\ a_P(y_j) = Z$$

$$a_Q : k[\mathrm{join}(P,Q)] \to k[R] \text{ such that } a_Q(x_i) = Z,\ a_Q(y_j) = g(y_j)$$

for all $i \in [0,n]$ and $j \in [0,m]$.

Note that a_P factors through the homomorphism $k[\mathrm{join}(P,Q)] \to k[\mathrm{join}(P,y_0)]$ such that $x_i \mapsto x_i$ and $y_j \mapsto y_0$ and the homomorphism $k[\mathrm{join}(P,y_0)] \to k[R]$ such that $x_i \mapsto f(x_i)$ and $y_0 \mapsto Z$. The first map is tame since it maps monomials to monomials and the second is tame since it is a free extension of the homomorphism f which is tame by assumption. Thus a_P is tame.

Similarly a_Q factors through the homomorphisms $k[\mathrm{join}(P,Q)] \to k[\mathrm{join}(x_0,Q)]$ such that $y_j \mapsto y_j$ and $x_i \mapsto x_0$ and $k[\mathrm{join}(x_0,Q)] \to k[R]$ such that $y_j \mapsto f(y_j)$ and $x_0 \mapsto Z$. These maps are tame as above and thus a_Q is a tame homomorphism.

But F is obtained by the Minkowski sum of a_P and a_Q since for all $v \in \mathrm{L}(\mathrm{join}(P,Q))$ we have

$$(a_P \star a_Q)(v) = a_P(v)a_Q(v)Z^{-1} = \begin{cases} f(v), & \text{if } v \in \mathrm{L}(P) \\ g(v) & \text{if } v \in \mathrm{L}(Q) \end{cases} = F(v)$$

Hence F is tame as desired. □

5 Multiples

In [3] any graded homomorphism from $k[c\Delta_n]$, where $c\Delta_n$ is the c^{th} multiple of the n-simplex, was shown to be tame. A similar argument as found in the proof of this theorem will be used to prove its generalization concerning the c^{th} multiple of a general lattice polytope, stated as Theorem 1(b). In order to prove this, we need the following lemma.

Lemma 2. *Suppose $\Phi : cP \to \mathbb{R}^l_+$ is an affine integral map. Then there exists a vector $v \in \mathbb{Z}^l_+$ and an affine integral map $\phi : P \to \mathbb{R}^l_+$ such that $\Phi = v + c\phi$.*

Proof of Lemma 2. Suppose $\mathrm{L}(P) = \{x_0,...,x_n\}$. Consider $\Phi(cx_i)$ where $x_i \in \mathrm{L}(P)$. Then $\Phi(cx_i) \in \mathbb{Z}^l_+$, say $\Phi(cx_i) = (a_{i1},...,a_{il})$ where $i \in [0,n]$.
Let $v = (\min\{a_{i1}\}_{i=1}^n, ..., \min\{a_{il}\}_{i=1}^n) \in \mathbb{Z}^l_+$. We will show that $\Phi(cx_i) - v$ are c^{th} multiples of integral vectors for all i. But for all $k \in [1,l]$ the k^{th} component of either $\Phi(cx_i) - v$ or $\Phi(cx_j) - v$, for some $j \neq i$, is 0. If k^{th} component of $\Phi(cx_i) - v$ is 0 this is a c^{th} multiple of an integer and we are done. So assume k^{th} component of $\Phi(cx_j) - v$ is 0 for some $j \neq i$. Since Φ is an affine integral map, $\Phi(cx_i) - \Phi(cx_j) = c(\Phi(x_i) - \Phi(x_j))$ is a c^{th} multiple of an integral vector. But $\Phi(cx_i) - \Phi(cx_j) = (\Phi(cx_i) - v) - (\Phi(cx_j) - v)$. Since the k^{th} component of $\Phi(cx_j) - v$ is 0, the k^{th} component of $\Phi(cx_i) - \Phi(cx_j)$ is the k^{th} component of $\Phi(cx_i) - v$, which is therefore a c^{th} multiple of an integer. Hence $\Phi(cx_i) - v$ are c^{th} multiples of integral vectors for all i. Denote $\frac{1}{c}(\Phi(cx_i) - v)$ by $\phi(x_i)$ for each i. Then we have an integral affine map $\phi : P \to \mathbb{R}^l_+$ such that $\Phi = v + c\phi$ as desired. □

Proof of Theorem 1(b). Let $\mathrm{L}(P) = \{x_0,...,x_n\}$. Let $f : k[cP] \to k[Q]$ be any graded homomorphism. Suppose $cP \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^m$. By a polytope change we can assume $\mathrm{L}(cP) \subset \mathbb{Z}^d_+$ and $\mathrm{L}(Q) \subset \mathbb{Z}^m_+$. Thus $\mathrm{L}(Q) \subset \{X_1^{a_1} \cdots X_m^{a_m} Z \mid a_i \geq 0\}$

By Section 3, $\varphi_x = fZ^{-1}$ gives rise to the affine map $\Phi : \mathrm{L}(cP) \to \mathbb{Z}^l_+$ which respects the binomial relations in $\mathrm{L}(cP)$. Thus by Lemma 2 there exists a vector $v \in \mathbb{Z}^l_+$ and an affine integral map $\phi : P \to \mathbb{R}^l_+$

such that $\Phi = v + c\phi$. Now, for all $x_i \in \mathrm{L}(P)$, $\phi(x_i) \in \mathbb{Z}_+^l$ and thus each $\phi(x_i)$ gives rise to a polynomial $\theta_i \in k[X_1, \dots, X_m]$ for each $i \in [0, 1]$. Similarly v gives rise to a polynomial $\psi \in k[X_1, \dots, X_m]$.
Let $x \in \mathrm{L}(cP)$. Without loss of generality we can assume $\mathrm{gp}(S(P)) = \mathbb{Z}_+^d$ (this follows from the fact that any d-dimensional polytope P in $\mathbb{R}^d$ is isomorphic as lattice polytopes to a polytope Q such that $\mathrm{gp}(S(Q)) = \mathbb{Z}^d$). Thus $x = a_0x_0 + \cdots + a_nx_n$ for some integers a_i such that $\sum_{i=0}^n a_i = c$. Then $\varphi_x = \psi\theta_0^{a_0} \cdots \theta_n^{a_n}$. Taking the c^{th} homothetic blow-up of

$$\Theta : k[P] \to k[Q] \textit{ such that } \Theta(x_i) = \theta_i Y$$

for all $x_i \in L(P)$, we see that f is obtained by a polytope change applied to $\Psi \star \Theta^c$ where

$$\Psi : k[cP] \to k[Q] \textit{ such that } \Psi(x) = \psi Y$$

for all $x \in L(cP)$.
Ψ is tame since it factors through the map $k[cP] \to k[t]$ such that $x \mapsto t$ for all $x \in cP$ and the map $k[t] \to k[Q]$ such that $t \mapsto \psi Y$.
Θ is tame by assumption since it is a homomorphism from $k[P]$. Thus we can conclude that f is tame. □

6 Segre Products

Proving Theorem 1(c) requires the most involved arguments of the results in this work and we consider it to be the main result.

The crucial step in proving the theorem is the lemma below. Note that $\mathrm{L}(P \times Q) = \mathrm{L}(P) \times L(Q)$. In the following, we will denote an element of $\mathrm{L}(P \times Q)$ by z_{ij} meaning that $z_{ij} = (x_i, y_j)$ for some $x_i \in \mathrm{L}(P)$ and $y_j \in \mathrm{L}(Q)$. Also note that, if $\mathrm{L}(P) = \{x_0, \dots, x_n\}$ and $\mathrm{L}(Q) = \{y_0, \dots, y_m\}$, we can identify P with its copy in $P \times Q$ having the lattice points $\{z_{00}, \dots, z_{n0}\}$ and identify Q with its copy having the lattice points $\{z_{00}, \dots, z_{0m}\}$. For a homomorphism f from $k[P \times Q]$ we will, for simplicity, denote $f(z_{i0})$ by $f(x_i)$, $f(z_{0j})$ by $f(y_j)$ and by $f(P)$ and $f(Q)$ we will mean $f(P \times \{y_0\})$ respectively $f(\{x_0\} \times Q)$.

Lemma 3. *Suppose P and Q are lattice polytopes. If $\alpha : \mathrm{L}(P \times Q) \to \mathbb{Z}_+^l$ is an affine map, then there exist a vector $b \in \mathbb{Z}_+^l$ and affine maps a_P, $a_Q : \mathrm{L}(P \times Q) \to \mathbb{Z}_+^l$ defined by*
$a_P(z_{ij}) = p_i$ and $a_Q(z_{ij}) = q_j$ for all $z_{ij} \in \mathrm{L}(P \times Q)$, for some p_i, $q_j \in \mathbb{Z}_+^l$, such that $\alpha = a_P + a_Q + b$.

Proof of Lemma 3. Suppose P and Q are lattice polytopes with $\mathrm{L}(P) = \{x_0, ..., x_n\}$ and $\mathrm{L}(Q) = \{y_0, ..., y_m\}$.
Let $\alpha : \mathrm{L}(P \times Q) \to \mathbb{Z}_+^l$ be an affine map.
Assume $\alpha(x_i) = (c_{i1}, ..., c_{il}) \in \mathbb{Z}_+^l$ for $i \in [0, n]$, and $\alpha(y_j) = (d_{j1}, ..., d_{jl}) \in \mathbb{Z}_+^l$ for $j \in [0, m]$.
Let $u_P = (\min\{c_{i1}\}_{i=0}^n, ..., \min\{c_{il}\}_{i=0}^n) \in \mathbb{Z}_+^l$. Denote $\alpha(x_i) - u_P$ by p_i.
Let $u_Q = (\min\{d_{j1}\}_{j=0}^m, ..., \min\{d_{jl}\}_{j=0}^m) \in \mathbb{Z}_+^l$. Denote $\alpha(y_j) - u_Q$ by q_j. Note that for each $k \in [1, l]$, the k^{th} component of p_i and q_j is 0 for some $i \in [0, n]$ respectively $j \in [0, m]$.
Define a_P and a_Q to be the affine maps

$$a_P : L(P \times Q) \to \mathbb{Z}_+^l \;\; \textit{such that} \;\; a_P(z_{ij}) = p_i \;\; \forall\, i \in [0, n],\, j \in [0, m]$$

$$a_Q : L(P \times Q) \to \mathbb{Z}_+^l \;\; \textit{such that} \;\; a_Q(z_{ij}) = q_j \;\; \forall\, j \in [0, m],\, i \in [0, n]$$

Note that the images of a_P and a_Q are integral translates of $\alpha(\mathrm{L}(P))$ respectively $\alpha(\mathrm{L}(Q))$. Denote these translates $\alpha(\mathrm{L}(P))'$ and $\alpha(\mathrm{L}(Q))'$.

Consider the map $\mathcal{L} : \mathbb{R}^l \to \mathbb{R}$ defined by $\boldsymbol{\xi} \mapsto \sum_{i=1}^l \xi_i$. Fix a point $x \in \alpha(\mathrm{L}(P))$. For an integral translate T of $\alpha(\mathrm{L}(P))$ let x^T be the copy of x in T. Define $\mathcal{L}(T) := \mathcal{L}(x^T)$. Thus $\mathcal{L}$ assigns a number to each translate of $\alpha(\mathrm{L}(P))$. Similarly fix a point $y \in \alpha(\mathrm{L}(Q))$ and define $\mathcal{L}(S) := \mathcal{L}(y^S)$ for any translate S of $\alpha(\mathrm{L}(Q))$ (where y^S is the copy of y in S).

Claim: Over all integral translates of $\alpha(\mathrm{L}(P))$ respectively $\alpha(\mathrm{L}(Q))$ in $\mathbb{Z}_+^l$, $\mathcal{L}$ attains its minimum value at $\alpha(\mathrm{L}(P))'$ respectively $\alpha(\mathrm{L}(Q))'$. Moreover, $\alpha(\mathrm{L}(P))'$ and $\alpha(\mathrm{L}(Q))'$ are unique with this property.

Suppose to the contrary that $\mathcal{L}(T) < \mathcal{L}(\alpha(\mathrm{L}(P))')$ for some translate $T \subset \mathbb{Z}_+^l$ of $\alpha(\mathrm{L}(P))$. Then $x_k^T < p_k$ for some $k \in [1,l]$ where p is the copy of x in $\alpha(\mathrm{L}(P))'$ (here x_k^T denotes the k^{th} component of x^T and similarly for p_k). Let $s \in \mathbb{Z}_+^l$ be the vector such that $s_k = 1$ and $s_j = 0$ for all $j \neq k$. Then $p - s \in \mathbb{Z}_+^l$ for all $p \in \alpha(\mathrm{L}(P))'$. But $p'_k = 0$ for some $p' \in \alpha(\mathrm{L}(P))'$ and thus $p' - s \notin \mathbb{Z}_+^l$, a contradiction.
Now suppose $\mathcal{L}(T) = \mathcal{L}(\alpha(\mathrm{L}(P))')$ for some translate $T \subset \mathbb{Z}_+^l$. By the argument above $x_k^T \geq p_k$ for all $k \in [1,l]$, and thus since $\mathcal{L}(x^T) = \mathcal{L}(p)$, we must have $x^T = p$ and hence $T = \alpha(\mathrm{L}(P))'$.
A similar argument shows that the claim is also true for $\alpha(\mathrm{L}(Q))'$.
Thus $\alpha(\mathrm{L}(P))'$ and $\alpha(\mathrm{L}(Q))'$ are the minimal copies of $\alpha(\mathrm{L}(P))$ and $\alpha(\mathrm{L}(Q))$ in $\mathbb{Z}_+^l$.

Case 1. (Non-degenerate) $\dim \alpha(\mathrm{L}(P \times Q)) = \dim \alpha(\mathrm{L}(P)) + \dim \alpha(\mathrm{L}(Q))$.
Let $k \in [1,l]$. Choose a vertex $v \in \alpha(\mathrm{L}(P \times Q))$ such that $v_k \leq w_k$ for all $w \in \alpha(\mathrm{L}(P \times Q))$. The vertex v corresponds to a unique pair $x \in \alpha(\mathrm{L}(P))$, $y \in \alpha(\mathrm{L}(Q))$ which in turn correspond to a unique pair $p_i \in \alpha(\mathrm{L}(P))'$ and $q_j \in \alpha(\mathrm{L}(Q))'$, i.e. $v = \alpha(z_{ij})$. It follows that $p_{ik} \leq p'_k$ for all $p' \in \alpha(\mathrm{L}(P))'$ and $q_{jk} \leq q'_k$ for all $q' \in \alpha(\mathrm{L}(Q))'$. But $p_k \geq 0$ for all $p \in \alpha(\mathrm{L}(P))'$ and $p_k = 0$ for some $p \in \alpha(\mathrm{L}(P))'$. Thus $p_{ik} = 0$. Similarly $q_{jk} = 0$. Thus $v_k \geq p_{ik} + q_{jk}$, i.e. $v_k = p_{ik} + q_{jk} + b_k$ where $b_k \in \mathbb{Z}_+$.

Now let $w \in \alpha(\mathrm{L}(P \times Q))$ be any other point. Then $w_k \geq v_k$, say $w_k - v_k = \delta$. Suppose $p_c \in \alpha(\mathrm{L}(P))'$ and $q_d \in \alpha(\mathrm{L}(Q))'$ are the points corresponding to w. Then $p_{ck} - p_{ik} = \delta_1 \geq 0$ and $q_{dk} - q_{jk} = \delta_2 \geq 0$ such that $\delta_1 + \delta_2 = \delta$.
Thus $w_k = v_k + \delta = p_{ik} + q_{jk} + \delta_1 + \delta_2 + b_k = p_{ck} + q_{dk} + b_k$.

Hence $v_k = p_{ik} + q_{jk} + b_k$ for all $k \in [1,l]$ i.e. $v = p_i + q_j + b$ for all $v \in \alpha(\mathrm{L}(P \times Q))$, where $b \in \mathbb{Z}_+^l$ is independent of v and p_i and q_j correspond to v. Thus $\alpha(z_{ij}) = a_P(z_{ij}) + a_Q(z_{ij}) + b$ as desired.

Case 2. (Degenerate) $\dim \alpha(\mathrm{L}(P \times Q)) < \dim \alpha(\mathrm{L}(P)) + \dim \alpha(\mathrm{L}(Q))$.
By perturbing the degenerate product into a non-degenerate we will prove the desired result by an approximation of Case 1.

Assume $\dim \alpha(\mathrm{L}(P \times Q)) = n \leq l$. Then $\alpha(\mathrm{L}(P)), \alpha(\mathrm{L}(Q)) \subset \mathbb{R}^n$. Let $e_1, \ldots, e_n$ be the standard basis vectors in $\mathbb{R}^n$.
Let $v_i = (0, e_i) \subset \mathbb{R}^{2n}$ for $i \in [1,n]$.
We can embed $\mathbb{R}^n$ into $\mathbb{R}^{2n}$ by letting $e_i \mapsto (e_i, 0) \in \mathbb{R}^{2n}$ for all $i \in [1,n]$.
Let $\lambda \in [0,1]$ be a real number. Define $w_{i\lambda} = (1-\lambda)e_i + \lambda v_i$, where we identify e_i with its copy in $\mathbb{R}^{2n}$ under the embedding above.
Define $E_\lambda = span\{w_1, \ldots, w_n\}$ for each λ. Note that $\dim E_\lambda = n$, $E_\lambda \to \mathbb{R}^n$ as $\lambda \to 0$, and $E_\lambda \cap \mathbb{R}^n = \{0\}$ for all $\lambda \in (0,1]$.

Consider the isomorphism $\mathbb{R}^n \to E_\lambda$ defined by $e_i \mapsto w_{i\lambda}$ for all $i \in [1,n]$. Let $\alpha(\mathrm{L}(P))_\lambda$ be the affine copy of $\alpha(\mathrm{L}(P))$ in E_λ under this isomorphism.
Then $\alpha(\mathrm{L}(P))_\lambda \to \alpha(\mathrm{L}(P))$ as $\lambda \to 0$. Also, $\alpha(\mathrm{L}(P))_\lambda \times \alpha(\mathrm{L}(Q)) \simeq \alpha(\mathrm{L}(P)) \times \alpha(\mathrm{L}(Q))$ and $\alpha(\mathrm{L}(P))_\lambda \times \alpha(\mathrm{L}(Q)) \to \alpha(\mathrm{L}(P \times Q))$ as $\lambda \to 0$.

As with $\alpha(\mathrm{L}(P))$, we can minimize each $\alpha(\mathrm{L}(P))_\lambda$ with respect to $E_\lambda^+ := \sum_{i=1}^n \mathbb{R}_+ w_{i\lambda}$. Denote the minimized copy of $\alpha(\mathrm{L}(P))_\lambda$ by $\alpha(\mathrm{L}(P))'_\lambda$. By the uniqueness of the minimized translates, $\alpha(\mathrm{L}(P))'_\lambda \to \alpha(\mathrm{L}(P))'$ as $\lambda \to 0$.

Let $v \in \alpha(\mathrm{L}(P \times Q))$. Let v^λ denote the copy of v in $\alpha(\mathrm{L}(P))_\lambda \times \alpha(\mathrm{L}(Q))$. Then $v^\lambda \to v$ as $\lambda \to 0$. Also, since $E_\lambda \cap \mathbb{R}^n = \{0\}$ for all $\lambda \in (0,1]$, $\alpha(\mathrm{L}(P))_\lambda \times \alpha(\mathrm{L}(Q))$ is non-degenerate for each $\lambda \in (0,1]$. Thus, by Case 1, there exists a unique pair $p_\lambda \in \alpha(\mathrm{L}(P))'_\lambda$, $q_\lambda \in \alpha(\mathrm{L}(Q))'$ corresponding to v_λ such that $v_\lambda = p_\lambda + q_\lambda + b_\lambda$, where $b_\lambda \in \mathbb{R}^{2n}$ is integral with respect to the basis $(e_1, \ldots, e_n, w_{1\lambda}, \ldots, w_{n\lambda})$ and is independent of v. But since $\alpha(\mathrm{L}(P))_\lambda \times \alpha(\mathrm{L}(Q)) \simeq \alpha(\mathrm{L}(P))_{\lambda'} \times \alpha(\mathrm{L}(Q))$ for all $\lambda, \lambda' \in [0,1]$, we have $q_\lambda \simeq q_{\lambda'}$. But $q_\lambda, q_{\lambda'} \in \alpha(\mathrm{L}(Q))$ and thus $q_\lambda = q_{\lambda'}$, i.e. q is independent of λ.
Thus $v_\lambda = p_\lambda + q + b_\lambda$. Since $p_\lambda \in \alpha(\mathrm{L}(P))'_\lambda \subset E_\lambda^+$ we can represent it as $p_\lambda = a_1 w_{1\lambda} + \cdots + a_n w_{n\lambda}$ where $a_i \in \mathbb{Z}_+$. Then $p_\lambda \to p = a_1 e_1 + \cdots + a_n e_n$ as $\lambda \to 0$, and $p \in \alpha(\mathrm{L}(P))'$ since $\alpha(\mathrm{L}(P))'_\lambda \to \alpha(\mathrm{L}(P))'$ as $\lambda \to 0$.
Also, since $b_\lambda = b_1 e_1 + \cdots b_n e_n + c_1 w_{1\lambda} + \cdots + c_n w_{n\lambda}$ for some $b_i, c_i \in \mathbb{Z}_+$, $b_\lambda \to b = b_1 e_1 + \cdots + b_n e_n \in \mathbb{Z}_+^n$.
Hence, since $lim_{\lambda \to 0} v_\lambda = lim_{\lambda \to 0} p_\lambda + q + lim_{\lambda \to 0} b_\lambda$, we have $v = p + q + b$ as desired. $\square$

Now we can prove the desired result.

Proof of Theorem 1(c). Suppose $P \subset \mathbb{R}^d$, $Q \subset \mathbb{R}^e$, and $R \subset \mathbb{R}^c$ are lattice polytopes. Let $f : k[P \times Q] \to k[R]$ be any graded homomorphism. By a polytope change we can assume $\mathrm{L}(P \times Q) = \mathrm{L}(P) \times \mathrm{L}(Q) \subset \mathbb{R}_+^{d+e}$ such that $\mathrm{L}(P) \cap \mathrm{L}(Q) = \{0\}$ and $\mathrm{L}(R) \subset \mathbb{R}_+^c$. Suppose $\mathrm{L}(P) = \{x_0, \dots, x_n\}$ and $\mathrm{L}(Q) = \{y_0, \dots, y_m\}$ and denote $\mathrm{L}(P \times Q) = \{z_{ij} | \, i \in [0, n], j \in [0, m]\}$ where z_{ij} corresponds to (x_i, y_j).

Since $k[R] \subset \{X_1^{a_1} \cdots X_c^{a_c} Z | \, a_i \geq 0\}$, the polynomials $\varphi(z_{ij}) = f(z_{ij})Z^{-1}$ belong to $k[X_1, \dots, X_c]$ and they respect the binomial relations in $L(P \times Q)$. Thus, by Section 3, φ gives rise to an affine map $\alpha : \mathrm{L}(P \times Q) \to \mathbb{Z}_+^l$ for some $l \in \mathbb{N}$. Thus, by Lemma 3, there exist a vector $b \in \mathbb{Z}_+^l$ and affine maps
$a_P : \mathrm{L}(P \times Q) \to \mathbb{Z}_+^l$ such that $a_P(z_{ij}) = p_i$ for all $z_{ij} \in \mathrm{L}(P \times Q)$,
$a_Q : \mathrm{L}(P \times Q) \to \mathbb{Z}_+^l$ such that $a_Q(z_{ij}) = q_j$ for all $z_{ij} \in \mathrm{L}(P \times Q)$
(for some p_i, $q_j \in \mathbb{Z}_+^l$), such that $\alpha = a_P + a_Q + b$.
Now, for all $z_{ij} \in \mathrm{L}(P \times Q)$, $a_P(z_{ij}) = p_i$ and $a_Q(z_{ij}) = q_j$ give rise to polynomials π_i, $\varrho_j \in k[X_1, \dots, X_c]$, respectively, for all $i \in [0, n]$ and $j \in [0, m]$. Similarly, $b \in \mathbb{Z}_+^l$ gives rise to the polynomial $\beta \in k[X_1, \dots, X_c]$. Then $\varphi(z_{ij}) = \pi_i \cdot \varrho_j \cdot \beta$ for all $z_{ij} \in L(P \times Q)$. But this means $f = (f_P \star f_Q) \star B$ where

$$f_P : k[P \times Q] \to k[R] \;\; \textit{such that} \;\; f_P(z_{ij}) = \pi_i Z$$

$$f_Q : k[P \times Q] \to k[R] \;\; \textit{such that} \;\; f_Q(z_{ij}) = \varrho_j Z$$

$$B : k[P \times Q] \to k[R] \;\; \textit{such that} \;\; B(z_{ij}) = \beta Z$$

for all $z_{ij} \in \mathrm{L}(P \times Q)$.
f_P is a tame homomorphism since it factors through the homomorphism $k[P \times Q] \to k[P]$ such that $z_{ij} \mapsto x_i$ for all j (which is tame since it maps monomials to monomials) and the homomorphism $k[P] \to k[R]$ such that $x_i \mapsto \pi_i Z$ for all i which is a homomorphism from $k[P]$ and thus tame.
Similarly f_Q is tame since it factors through the map $k[P \times Q] \to k[Q]$ such that $z_{ij} \mapsto y_j$ for all i (tame as above) and the map $k[Q] \to k[R]$ such that $y_j \mapsto \varrho_j Z$ for all j which is a homomorphism from $k[Q]$ and is thus tame.
Finally, B factors through the homomorphism $k[P \times Q] \to k[t]$ such that $z_{ij} \mapsto t$ for all $z_{ij} \in \mathrm{L}(P \times Q)$, and the homomorphism $k[t] \to k[R]$ such that $t \mapsto \beta Z$. The first factor is tame since it maps monomials to monomials and the second is tame since it is a free extension of the map $k \to k$.
Therefore f is a tame homomorphism, as desired. $\square$

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